Ensembles de Lattès-Julia

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Who am I? After a thesis at the Mathematics Section of the University of Geneva and an academic career of about ten years in the field of string theory, I now work in industry at G-Research, a company using advanced techniques in statistics and machine learning to predict financial markets.

Context of the algorithm used. The two images presented are constructed by means of conformal mappings of the sphere to itself. By *mapping*, we mean a way to glue a sphere on another, assuming it is made of a material infinitely extensible but impossible to tear (continuity). A mapping is said to be *conformal* if it deforms the immediate environment of each point by means of rotation and expansion/contraction, without stretching it in a privileged direction. Given a point on the sphere, the mapping can be made to act on it to obtain a new point. By iterating this process, we obtain a sequence of points on the sphere. One can then color each point of the sphere according to certain characteristics of its orbit, for example the average distance of the points of the orbit to the north pole of the sphere. For certain well-chosen conformal mappings, one obtains images such as those presented here (which represent portions of spheres).

It is interesting to observe the behavior of the orbits of the points in a small neighborhood of a given point. For some points, the orbits of the neighborhood remain close to each other. For others, they diverge after a sufficient number of iterations: they are called *chaotic*. The set of points with chaotic orbits is the *Julia set* of the application. The conformal mappings represented here have the particularity of having a dense Julia set, which essentially means that all points have chaotic orbits.

The fractal and self-similar structures that can be observed on these images are the direct consequences of the fact that the underlying mapping is conformal, and therefore the immediate environment of each point is not stretched. A non-conformal mapping would produce structures so stretched that they would become unrecognizable. The fact that the application has a dense Julia set results in fractal structures appearing all over the image: no region of the image is uniformly colored.

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But what is it useful for? Conformal geometry is the branch of mathematics that studies spaces in which distances cannot necessarily be measured, but in which angles can be measured in any case. It has many applications in mathematics, and perhaps most surprisingly in physics. Some physical systems, for certain so-called "critical" values of their parameters (for example temperature), exhibit the same type of scale invariance and self-similarity that can be observed in these images. They are described by physical theories which include conformal mappings among their symmetries. These conformal theories are also at the heart of modern research in string theory and quantum gravity.

I want more details! We have not yet specified the type of conformal mappings used to create these images. These are Lattès maps, which are built as follows. Let us consider a 2-dimensional torus (the shape of a doughnut, or a bicycle inner tube). The torus admits very simple conformal mappings, which are simply rotations (for example the rotation of the inner tube due to the rotation of the bicycle wheel). The existence of certain conformal mappings from the torus to the sphere (called ramified coverings) allows the transfer of these simple conformal mappings of the torus to conformal mappings of the sphere. These are the Lattès maps, described by the French mathematician Samuel Lattès in 1918.

For more details:

- A blog article explaining in more detail the construction of Lattès' maps: http://www.algw.net/blog/blog.php?Post=20130428
- An excellent article by John Milnor on Lattès maps: https://arxiv.org/abs/math/0402147.
- For more images of the same type, visit http://www.algw.net