Chord diagrams with 5 chords.

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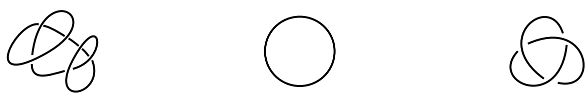
Who am I? I'm a PhD student at the University of geneva. The picture you see is a byproduct of my doctoral project in the domain of mathematical physics. One of the big problems in the field is extending classical systems to their quantum versions, and these circles with chords play an important role in some variants of this problem.

Context of the algorithm used. The diagrams you see are called chord diagrams on a circle. Chords are those dashed lines connecting bullets lying on the solid circle. The picture represents all[1] chord diagrams with 5 chords.

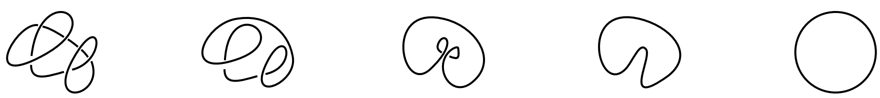
The main function of my algorithm is studying relations between those chord diagrams. These relations, coming from knot theory, make it possible to express all of these 372 diagrams in terms of just 10 of them. For more details, read on.

But what is it useful for? Chords diagrams come from knot theory, a beautiful mathematical subject. Some of its recent applications include quantum computing, DNA and other objects which can become "tangled".

Knot theory studies knots made of one string with ends tied together:



Immediately, you see that the first knot can be deformed to the second one: mathematicians say that it can be "unknotted".



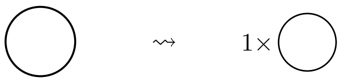
It is less obvious that the third knot cannot be unknotted. To distinguish knots that cannot be deformed into each other, mathematicians found many

[1]For optimization purposes, some of the possible chord diagrams are not present. This is related to the fact that the shortest chords  are always on the right side of the triangle.

knot invariants, machines that take a knot and give you back something (a number, a polynomial, a sum of diagrams...). These machines have to satisfy an important property: if we deform a knot, the output of the machine does not change. Thus, if we have two knots with different knot invariants, we know they cannot be deformed into each other.[2]

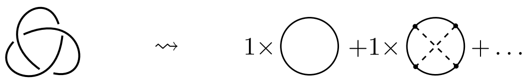
One of the important knot invariants is the Kontsevich integral, which brings us back to our chord diagrams. The Kontsevich integral takes a knot and gives you back a sum of chord diagrams, each multiplied by some number.

For example, for the unknotted not, we get



i.e. just one chord diagram with zero chords (multiplied by 1).

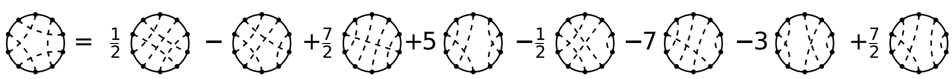
However, for the third knot from our list, we get



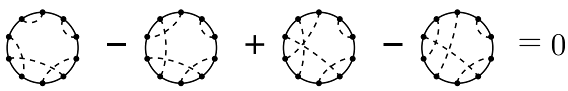
Because the results are different, we now know that the last knot cannot be unknotted.

I want more details ! The Kontsevich integral can, in general, change if we deform the underlying knot. We thus *define* the different chord diagrams, corresponding to the same knot, to be equal. This may feel like cheating, but luckily, we have a simple method, called *4T relations*, to tell us which chord diagrams we need to identify.

A simple consequence of this identification is that two chord diagrams are equal if they are related by rotation. However, we also get more complicated identities, for example



The purpose of my algorithm is finding all such identities. The *4T relations* mentioned above give in total 3294 relations between chord diagrams of the form



[2] However, if the knot invariants are equal, we don't know if the knots are related by deformations. Knot invariants satisfying this property are called *complete*. It is a difficult unsolved problem whether the Kontsevich integral is complete.

After simplifying these 3924 equations for the 372 chord diagrams, we finally get that only 10 of the chord diagrams are independent, all the remaining 362 can be expressed using 10 chord diagrams. More abstractly, we say that the space of chord diagrams with 4T relations has dimension 10.

Finally, let me explain how this relates to quantization, mentioned at the beginning. In the quantization problem I was looking at, the *quantum corrections* can be expressed using these chord diagrams. To get a consistent quantum system, these corrections cannot be chosen arbitrarily, but they need to satisfy a condition called *associativity*. However, it is possible to rewrite this condition in terms of knots, whose Kontsevich integral gives the chord diagrams describing the quantization. Checking the associativity condition, which is an infinite system of complicated equations, thus simplifies to checking a simple equivalence of knots.

For more information, I recommend the book "Introduction to Vassiliev Knot Invariants" by Chmutov, Duzhin and Mostovoy, available at `https://arxiv.org/abs/1103.5628`