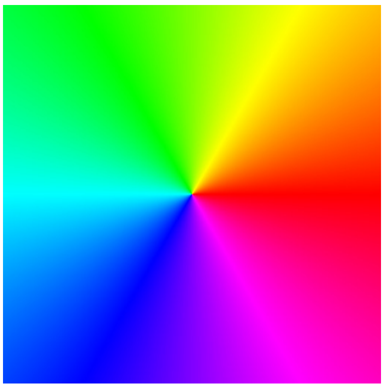
Fly me to Tan (moon) & Come wash with me

Felix Günther,

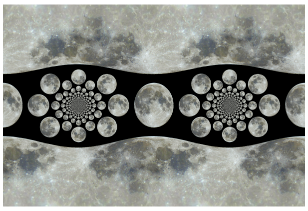
Who am I? I am a mathematician studying discretizations of classical geometrical theories, including complex analysis. In my PhD thesis, I developed a discrete theory of complex analysis that shares many notions and theorems with the classical theory. And even the proofs work in an almost analogous way! In 2017, the PRIME fellowship of the German Academic Exchange Service DAAD enabled me to continue my research on a discrete theory of holomorphic spinors at the Université de Genève. In addition to my work as a researcher, I am an active science communicator and regularly present mathematics and my research to broad audiences.

Context of the algorithm used. The underlying technique to produce such images is called domain colouring. Domain colouring is a very popular method to visualize complex functions and to picture four dimensions! Normally, a plane of points of rainbow colours and different brightness is chosen as a reference image. The hue indicates the angle, the brightness the distance to the centre.



But mathematics becomes more beautiful if you take an everyday picture and periodically continue it both in the horizontal and in the vertical direction! This grid of pictures is then identified with the complex plane. Given a function f, the point z on the complex plane is now coloured by the colour of $f(z)$ in the grid. For example, the centre of the moon is shown whenever

$tan(z)$ is equal to zero (or another integral point of the grid that is used), so whenever z is equal to an integer multiple of π. If you consider the left picture on Come wash with me and the function $f(z) = z^3$, any three points that have the same distance to the centre and have a 120 degree angle in between get the same colour.



If you did not like curve sketching of real functions in your high school mathematics course, you may now love curve sketching of complex functions!

But what is it useful for?

Holomorphic functions are special maps of the (complex) plane to itself that are generically angle-preserving. This sounds more complicated than it is, you probably come across such maps every week. In fact, web mapping services are using holomorphic maps to show us where you are. Considering the earth as a compactified complex plane (the so-called Riemann sphere), a chart is nothing else than a map from the Riemann sphere to the complex plane. The historical reason why angle-preserving maps became so popular is that they were crucial for sailors to navigate their ships. Hundreds of years ago, the only way to determine your position on the open sea was to measure angles and to compare them with your chart.

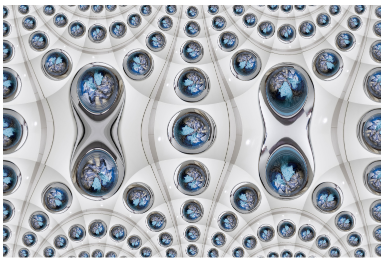
I want more details! Since these maps preserve angles, they locally preserve shapes. That is why you can still recognize the original picture of the moon onto which the complex tangent function is applied. But the distortion gives you a new and fascinating perspective on the moon and the infinity of the sky, or of the washing machine and its spin cycle. Conversely, the distorted pictures give you more insight into the mathematical behaviour of the corresponding function.

Fly me to tan(moon)

The tangent function is holomorphic apart from its poles where it takes the value infinity. To the left and to the right of the centre, at $\pm\frac{\pi}{2}$, you can clearly spot these poles: the moon appears there infinitely many times. You also observe that the moon appears almost not distorted at the centre. This corresponds to the asymptotic behaviour $\tan(z) \sim z$ near zero, a formula that you may remember from your physics class in high school. The more you look to the top or to the bottom of the picture, the less variation you can see. This actually corresponds to the property $\tan(x \pm i) \to \pm i$ if $y \to \infty$.

Come wash with me

The left picture shows you the function $f(z) = z^3$. You clearly observe its threefold symmetry. It preserves the shapes of the bed linen almost everywhere; but at zero, the light blue pillowcase wears out. This is called a ramification point and it always occurs when the derivative of the function is zero, as it is the case for $f'(z) = 3z^2$. Since the zero is of order 2, all the linen occurs 2+1=3 times around the centre. Have you ever experienced the situation that you got more out of the washing machine that you have put inside? Mathematics makes it happen! The central picture shows you the function $f(z) = \frac{1}{z}$ and infinitely many washing machines near the centre. That is because the function takes the value infinity at zero, which is called a pole of the function. The function is then holomorphic apart from this point. But which function is shown on the right? Have a close look and make your guess before you continue reading any further!



You observe that the function is periodic in the horizontal direction, say 2π-periodic. You see ramification points to the left and to the right from the centre at $\pm\frac{pi}{2}$, hence the depicted function is equal to zero at $\pm\frac{pi}{2}$. Finally, $f(z) \sim z$ near the centre. So do you know now which function is applied to the washing machine? It is the sine function! The sine function is 2π-periodic, its derivative is the cosine function that is equal to zero at $\pm\frac{pi}{2}$, and you may remember the approximation $\sin(z) \sim z$ near zero from your physics class in high school.