Diffusion-limited aggregation - Amanda Turner

Who am I? Amanda Turner is a Senior Lecturer in Mathematics and Statistics at Lancaster University, UK, who was a Visiting Professor at the University of Geneva in 2018/20. Her research interests lie at the interface of probability theory, complex analysis and mathematical physics. She is particularly interested in understanding the long-time behaviour of planar random growth processes which arise in physical settings, such as cell growth and polymer creation.

But what is it useful for ? Random growth clusters in the physical world usually consist of a large number of particles, each of which is small relative to the size of the cluster. Although the randomness typically occurs at a microscopic level through the attachment rule for each successive particle, we observe the clusters at a macroscopic level where we cannot see the individual particles. Random growth processes are completely unpredictable at the level of particles, however large clusters often exhibit predictable or 'universal' behaviour. The aim of studying mathematical models is to extract the principle mechanisms underlying this universal behaviour. As random growth models can be difficult to analyse mathematically, computer simulations are used to study properties such as the growth rate or the fractal dimension of the cluster. For example, it is predicted using computer simulation that diffusion-limited aggregation has a fractal dimension of 1.71.

In 1998, physicists Hastings and Levitov devised an approach to modelling planar growth in which they represented growing clusters as compositions of special types of functions, called conformal mappings. These are functions from the complex plane to itself which locally preserve angles between lines. This approach provides a way in which techniques from complex analysis can be used to study planar random growth.

I want more details ! In the simulation, we represent each particle as a small slit (line segment). It is possible to write down an explicit conformal mapping which takes the exterior unit disk $\{|z| > 1\}$ to the exterior of the unit disk with a slit of length d removed $\{|z| > 1\} \setminus (1, d]$. This mapping is used as a representation of a particle of length d attached to the unit disk at position 1 (Figure 1). The mapping can be rotated to represent a slit attached at position $e^{i\theta}$ for any angle $\theta \in [0, 2\pi)$.

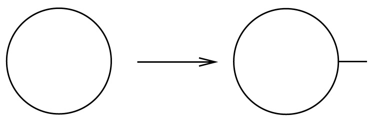

Figure 1: The conformal mapping corresponding to a single slit particle attached at 1.

The computer simulation constructs a cluster as follows. Step-by-step it generates a sequence of angles $\Theta_1, \Theta_2, \dots \in [0, 2\pi)$ and lengths $d_1, d_2, \dots > 0$ according to some rule which depends on the precise physical model which is to be constructed. For each $n \in \mathbb{N}$, let F_n denote the conformal mapping corresponding to a slit of length d_n attached at angle Θ_n. A sequence of conformal bijections $\Phi_n : \{|z| > 1\} \to \{|z| > 1\} \setminus K_n$ is constructed by setting $\Phi_0(z) = z$ and recursively defining

$$\Phi_n(z) = \Phi_{n-1} \circ F_n(z) = F_1 \circ \cdots \circ F_n(z).$$

Note that $K_0 = \{|z| \leq 1\}$ and $K_n = K_{n-1} \cup \Phi_{n-1}(e^{i\Theta_n}(1, 1 + d_n])$, so the sequence $K_0 \subset K_1 \subset K_2 \subset \cdots$ represents a growing cluster where the unit disk K_0 is the seed particle and at time n the particle $P_n = \Phi_{n-1}(e^{i\Theta_n}(1, 1 + d_n])$ is added to the cluster (Figure 2). A cluster consisting of n particles is then given by the image of the circle $\{|z| = 1\}$ under the map Φ_n.

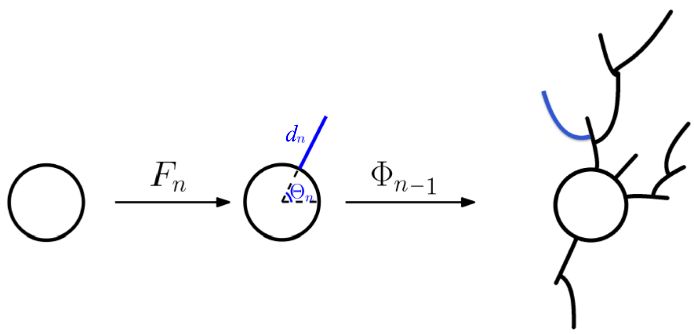

Figure 2: Diagram illustrating how growing clusters can be constructed as compositions of conformal mappings.